Chapter 1

Playing the Numbers Game

In This Chapter

- Finding out how numbers were invented
- Looking at a few familiar number sequences
- Examining the number line
- Understanding four important sets of numbers

One of the useful things about numbers is that they're *conceptual,* which means that in an important sense, they're all in your head. (This fact, however, probably won't get you out of having to know about them — nice try!)

For example, you can picture three of anything: three cats, three baseballs, three cannibals, three planets. But just try to picture the concept of three all by itself, and you find it's impossible. Oh, sure, you can picture the numeral 3, but the *threeness* itself — much like love or beauty or honor — is beyond direct understanding. But after you have the *concept* of three (or four, or a million), you have access to an incredibly powerful system for understanding the world: mathematics.

In this chapter, I give you a brief history of how numbers came into being. I discuss a few common *number sequences* and show you how these connect with simple math *operations* like addition, subtraction, multiplication, and division.

After that, I describe how some of these ideas come together with a simple yet powerful tool – the *number line*. I discuss how numbers are arranged on the number line, and I also show you how to use the number line as a calculator for simple arithmetic.

Finally, I describe how the *counting numbers* (1, 2, 3, . . .) sparked the invention of more unusual types of numbers such as *negative numbers*, *fractions*, and *irrational numbers*. I also show you how these *sets of numbers* are *nested* — that is, how one set of numbers fits inside another, which fits inside another.

Inventing Numbers

Historians believe that the first number systems came into being at the same time as agriculture and commerce. Before that, people in prehistoric, hunter-gatherer societies were pretty much content to identify bunches of things as “a lot” or “a little.”

But as farming developed and trade between communities began, more precision was needed. So people began using stones, clay tokens, and similar objects to keep track of their goats, sheep, oil, grain, or whatever commodity they had. These tokens could be exchanged for the objects they represented in a one-to-one exchange.

Eventually, traders realized that they could draw pictures instead of using tokens. Those pictures evolved into tally marks and, in time, into more complex systems. Whether they realized it or not, their attempts to keep track of commodities had led these early humans to invent something entirely new: *numbers.*

Throughout the ages, the Babylonians, Egyptians, Greeks, Romans, Mayans, Arabs, and Chinese (to name just a few) all developed their own systems of writing numbers.

Although Roman numerals gained wide currency as the Roman Empire expanded throughout Europe and parts of Asia and Africa, the more advanced system that the Arabs invented turned out to be more useful. Our own number system, the Hindu-Arabic numbers (also called decimal numbers), is closely derived from these early Arabic numbers.

Understanding Number Sequences

Although numbers were invented for counting commodities, as I explain in the preceding section, they were soon put to a wide range of applications. Numbers could be useful for measuring distances, counting money, amassing an army, levying taxes, building pyramids, and lots more.

But beyond their many uses for understanding the external world, numbers also have an internal order all their own. So, numbers are not only an *invention,* but also equally a *discovery:* a landscape that seems to exist independently, with its own structure, mysteries, and even perils.

One path into this new and often strange world is the *number sequence:* an arrangement of numbers according to a rule. In the following sections I introduce you to a variety of number sequences that are useful for making sense of numbers.

Evening the odds

One of the first things you probably heard about numbers is that all of them are either even or odd. For example, you can split an even number of marbles *evenly* into two equal piles. But when you try to divide an odd number of marbles the same way, you always have one *odd,* leftover marble. Here are the first few even numbers:

2 4 6 8 10 12 14 16 ...

You can easily keep the sequence of even numbers going as long as you like. Starting with the number 2, keep adding 2 to get the next number.

Similarly, here are the first few odd numbers:

1 3 5 7 9 11 13 15 ...

The sequence of odd numbers is just as simple to generate. Starting with the number 1, keep adding 2 to get the next number.

Patterns of even or odd numbers are the simplest number patterns around, which is why kids often figure out the difference between even and odd numbers soon after learning to count.

Counting by threes, fours, fives, and so on

After you get used to the concept of counting by numbers greater than one, you can run with it. For example, here's what counting by threes looks like:

3 6 9 12 15 18 21 24 ...

This time, the pattern is generated by starting with 3 and continuing to add 3.

Similarly, here's how to count by fours:

4 8 12 16 20 24 28 32 ...

And here's how to count by fives:

5 10 15 20 25 30 35 40 ...

Counting by a given number is a good way to begin learning the multiplication table for that number, especially for the numbers you're kind of sketchy on. (In general, people seem to have the most trouble multiplying by 7, but 8 and 9 are also unpopular.) In Chapter 3, I show you a few tricks for memorizing the multiplication table once and for all.

These types of sequences are also useful for understanding factors and multiples, which you get a look at in Chapter 8.

Getting square with square numbers

When you study math, sooner or later you probably want to use visual aids to help you see what numbers are telling you. (Later in this book, I show you how one picture can be worth a thousand numbers when I discuss geometry in Chapter 16 and graphing in Chapter 17.)

The tastiest visual aids you'll ever find are those little square cheese-flavored crackers. (You probably have a box sitting somewhere in the pantry. If not, saltine crackers or any other square food works just as well.) Shake a bunch out of a box and place the little squares together to make bigger squares. Figure 1-1 shows the first few.

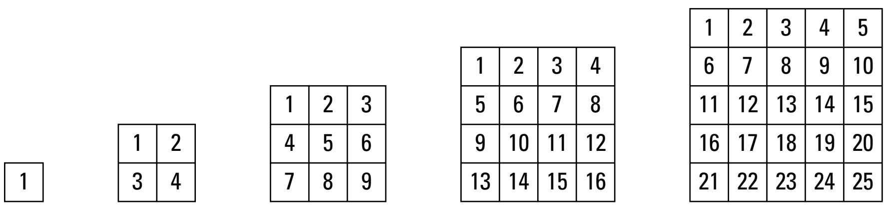


Figure 1-1: Square numbers.

Voila! The square numbers:

1 4 9 16 25 36 49 64 ...

You get a *square number* by multiplying a number by itself, so knowing the square numbers is another handy way to remember part of the multiplication table. Although you probably remember without help that $2 \times 2 = 4$, you may be sketchy on some of the higher numbers, such as $7 \times 7 = 49$. Knowing the square numbers gives you another way to etch that multiplication table forever into your brain, as I show you in Chapter 3.

Square numbers are also a great first step on the way to understanding exponents, which I introduce later in this chapter and explain in more detail in Chapter 4.

Composing yourself with composite numbers

Some numbers can be placed in rectangular patterns. Mathematicians probably should call numbers like these "rectangular numbers," but instead they chose the term *composite numbers*. For example, 12 is a composite number because you can place 12 objects in rectangles of two different shapes, as shown in Figure 1-2.

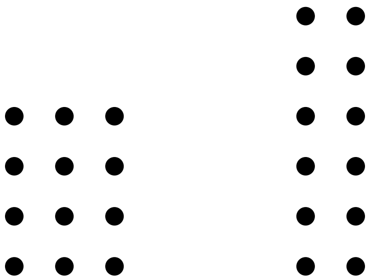


Figure 1-2: The number 12 laid out in two rectangular patterns.

As with square numbers, arranging numbers in visual patterns like this tells you something about how multiplication works. In this case, by counting the sides of both rectangles, you find out the following:

$3 \times 4 = 12$

$2 \times 6 = 12$

Similarly, other numbers such as 8 and 15 can also be arranged in rectangles, as shown in Figure 1-3.

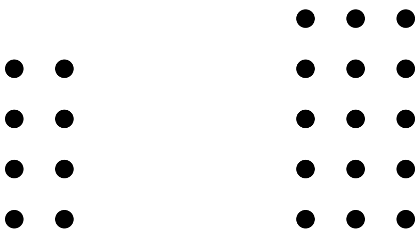


Figure 1-3: Composite numbers, such as 8 and 15, can form rectangles.

As you can see, both of these numbers are quite happy being placed in boxes with at least two rows and two columns. And these visual patterns show this:

$2 \times 4 = 8$

$3 \times 5 = 15$

The word *composite* means that these numbers are *composed of* smaller numbers. For example, the number 15 is composed of 3 and 5 — that is, when you multiply these two smaller numbers, you get 15. Here are all the composite numbers between 1 and 16:

4 6 8 9 10 12 14 15 16

Notice that all the square numbers (see "Getting square with square numbers") also count as composite numbers because you can arrange them in boxes with at least two rows and two columns. Additionally, lots of other nonsquare numbers are also composite numbers.

Stepping out of the box with prime numbers

Some numbers are stubborn. Like certain people you may know, these numbers — called *prime numbers* — resist being placed in any sort of a box. Look at how the number 13 is depicted in Figure 1-4, for example.

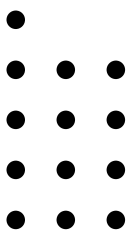


Figure 1-4: Unlucky 13, a prime example of a number that refuses to fit in a box.

Try as you may, you just can't make a rectangle out of 13 objects. (That may be one reason the number 13 got a bad reputation as unlucky.) Here are all the prime numbers less than 20:

2 3 5 7 11 13 17 19

As you can see, the list of prime numbers fills the gaps left by the composite numbers (see the preceding section). Therefore, every counting number is either prime or composite. The only exception is the number 1, which is neither prime nor composite. In Chapter 8, I give you a lot more information about prime numbers and show you how to *decompose* a number — that is, break a composite number down to its prime factors.

Multiplying quickly with exponents

Here's an old question that still causes surprises: Suppose you took a job that paid you just 1 penny the first day, 2 pennies the second day, 4 pennies the third day, and so on, doubling the amount every day, like this:

1 2 4 8 16 32 64 128 256 512 ...

As you can see, in the first ten days of work, you would've earned a little more than $10 (actually, $10.23 — but who's counting?). How much would you earn in 30 days? Your answer may well be, "I wouldn't take a lousy job like that in the first place." At first glance, this looks like a good answer, but here's a glimpse at your second ten days' earnings:

... 1,024 2,048 4,096 8,192 16,384 32,768 65,536 131,072 262,144 524,288 ...

By the end of the second 10 days, your total earnings would be over $10,000. And by the end of the third week, your earnings would top out around $10,000,000! How does this happen? Through the magic of exponents (also called *powers*). Each new number in the sequence is obtained by multiplying the previous number by 2:

$2^1 = 2 = 2$

$2^2 = 2 \times 2 = 4$

$2^3 = 2 \times 2 \times 2 = 8$

$2^4 = 2 \times 2 \times 2 \times 2 = 16$

As you can see, the notation 2^4 means *multiply 2 by itself 4 times.*

You can use exponents on numbers other than 2. Here's another sequence you may be familiar with:

1 10 100 1,000 10,000 100,000 1,000,000...

In this sequence, every number is 10 times greater than the number before it. You can also generate these numbers using exponents:

$10^1 = 10 = 10$

$10^2 = 10 \times 10 = 100$

$10^3 = 10 \times 10 \times 10 = 1{,}000$

$10^4 = 10 \times 10 \times 10 \times 10 = 10{,}000$

This sequence is important for defining *place value*, the basis of the decimal number system, which I discuss in Chapter 2. It also shows up when I discuss decimals in Chapter 11 and scientific notation in Chapter 15. You find out more about exponents in Chapter 5.

Looking at the Number Line

As kids outgrow counting on their fingers (and use them only when trying to remember the names of all seven dwarfs), teachers often substitute a picture of the first ten numbers in order, like the one shown in Figure 1-5.

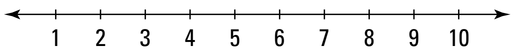


Figure 1-5: Basic number line.

This way of organizing numbers is called the *number line*. People often see their first number line — usually made of brightly colored construction paper — pasted above the blackboard in school. The basic number line provides a visual image of the *counting numbers* (also called the *natural numbers*), the numbers greater than 0. You can use it to show how numbers get bigger in one direction and smaller in the other.

In this section, I show you how to use the number line to understand a few basic but important ideas about numbers.

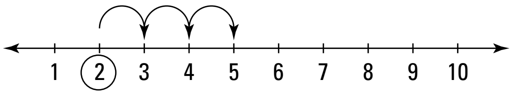
Adding and subtracting on the number line

You can use the number line to demonstrate simple addition and subtraction. These first steps in math become a lot more concrete with a visual aid. Here's the main thing to remember:

- As you go *right,* the numbers go *up,* which is *addition* (+).
- As you go *left,* the numbers go *down,* which is *subtraction* (–).

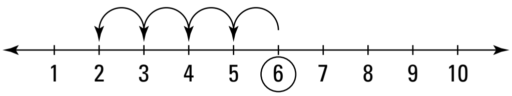
For example, 2 + 3 means you *start at* 2 and *jump up* 3 spaces to 5, as illustrated in Figure 1-6.

Figure 1-6: Moving through the number line from left to right.



As another example, 6 – 4 means *start at* 6 and *jump down* 4 spaces to 2. That is, 6 – 4 = 2. See Figure 1-7.

Figure 1-7: Moving through the number line from right to left.

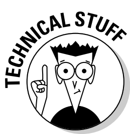


You can use these simple up and down rules repeatedly to solve a longer string of added and subtracted numbers. For example, 3 + 1 – 2 + 4 – 3 – 2 means 3, *up* 1, *down* 2, *up* 4, *down* 3, and *down* 2. In this case, the number line would show you that 3 + 1 – 2 + 4 – 3 – 2 = 1.

I discuss addition and subtraction in greater detail in Chapter 3.

Getting a handle on nothing, or zero

An important addition to the number line is the number 0, which means *nothing, zilch, nada.* Step back a moment and consider the bizarre concept of nothing. For one thing — as more than one philosopher has pointed out — by definition, *nothing* doesn't exist! Yet, we routinely label it with the number 0, as shown in Figure 1-8.



Actually, mathematicians have an even more precise labeling of *nothing* than zero. It's called the *empty set,* which is sort of the mathematical version of a box containing nothing. I introduce you to some basic set theory in Chapter 20.

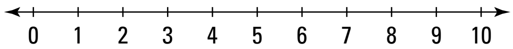


Figure 1-8: The number line starting at 0 and continuing with 1, 2, 3, . . . 10.

Nothing sure is a heavy trip to lay on little kids, but they don't seem to mind. They understand quickly that when you have three toy trucks and someone else takes away all three of them, you're left with zero trucks. That is, 3 – 3 = 0. Or, placing this on the number line, 3 – 3 means start at 3 and go down 3, as shown in Figure 1-9.

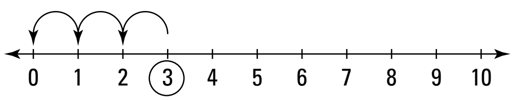


Figure 1-9: Starting at 3 and moving down three.

In Chapter 2, I show you the importance of 0 as a *placeholder* in numbers and discuss how *leading zeros* can be attached to a number without changing its value.

Infinity: Imagining a never-ending story

The arrows at the ends of the number line point onward to a place called *infinity,* which isn't really a place at all, just the idea of *foreverness,* because the numbers go on forever. But what about a million billion trillion quadrillion — do the numbers go even higher than that? The answer is yes, because for any number you name, you can add 1 to it.

The wacky symbol ∞ represents infinity. Remember, though, that ∞ isn't really a number but the *idea* that the numbers go on forever.

Because ∞ isn't a number, you can't technically add the number 1 to it, any more than you can add the number 1 to a cup of coffee or your Aunt Louise. But even if you could, ∞ + 1 would equal ∞.

Taking a negative turn: Negative numbers

When people first find out about subtraction, they often hear that you can't take away more than you have. For example, if you have four pencils, you can take away one, two, three, or even all four of them, but you can't take away more than that.

It isn't long, though, before you find out what any credit card holder knows only too well: You can, indeed, take away more than you have — the result is a *negative number*. For example, if you have \$4 and you owe your friend \$7, you're \$3 in debt. That is, $4 - 7 = -3$. The minus sign in front of the 3 means that the number of dollars you have is three less than 0. Figure 1-10 shows how you place negative whole numbers on the number line.

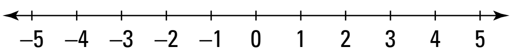


Figure 1-10: Negative whole numbers on the number line.

Adding and subtracting on the number line works pretty much the same with negative numbers as with positive numbers. For example, Figure 1-11 shows how to subtract 4 – 7 on the number line.

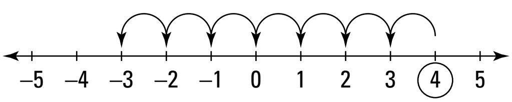


Figure 1-11: Subtracting 4 – 7 on the number line.

You find out all about working with negative numbers in Chapter 4.

TIP

Placing 0 and the negative counting numbers on the number line expands the set of counting numbers to the set of *integers*. I discuss the integers in further detail later in this chapter.

Multiplying the possibilities

Suppose you start at 0 and circle every other number on a number line, as shown in Figure 1-12. As you can see, all the even numbers are now circled. In other words, you've circled all the *multiples of two*. (You can find out more about multiples in Chapter 8.) You can now use this number line to multiply any number by two. For example, suppose you want to multiply 5 × 2. Just start at 0 and jump 5 circled spaces to the right.

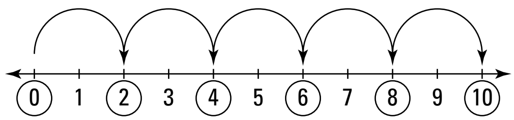


Figure 1-12: Multiplying 5 × 2 using the number line.

This number line shows you that 5 × 2 = 10.

Similarly, to multiply –3 × 2, start at 0 and jump 3 circled spaces to the left (that is, in the negative direction). Figure 1-13 shows you that –3 × 2 = –6. What's more, you can now see why multiplying a negative number by a positive number always gives you a negative result. (I talk about multiplying by negative numbers in Chapter 4.)

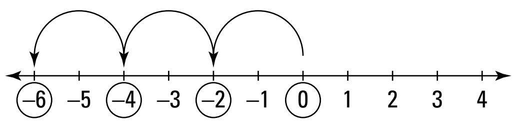


Figure 1-13: –3 × 2 = –6, as depicted on the number line.

Multiplying on the number line works no matter what number you count off by. For example, in Figure 1-14 I count off by 5s.

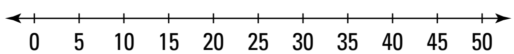


Figure 1-14: Number line counted off by 5s.

This time, the only numbers I've marked are the *multiples of 5,* so I can use this number line to multiply any number by 5. For example, Figure 1-15 shows how to multiply 2 × 5.

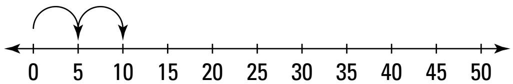


Figure 1-15: Multiplying 2 × 5 with the number line.

So 2 × 5 = 10, the same result as when you multiply 5 × 2. This result is an example of the *commutative property of multiplication* — you can reverse the order of any multiplication problem and still get the same answer. (I discuss the commutative property in Chapter 4.)

Dividing things up

You can also use the number line to divide. For example, suppose you want to divide 6 by some other number. First, draw a number line that begins at 0 and ends at 6, as in Figure 1-16.

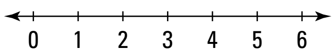


Figure 1-16: Number line from 0 to 6.

Now, to find the answer to 6 ÷ 2, just split this number line into two equal parts, as shown in Figure 1-17. This split (or *division*) occurs at 3, showing you that 6 ÷ 2 = 3.

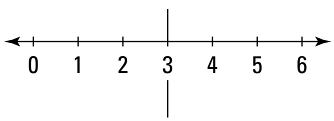
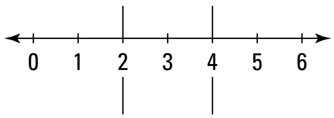


Figure 1-17: Getting the answer to 6 ÷ 2 by splitting the number line.

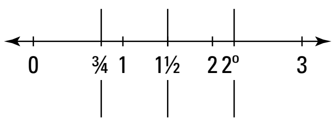
Similarly, to divide 6 ÷ 3, split the same number line into three equal parts, as in Figure 1-18. This time you have two splits, so use the one closest to 0. This number line shows you that 6 ÷ 3 = 2.

Figure 1-18: Dividing 6 ÷ 3 with the number line.



But suppose you want to use the number line to divide a small number by a larger number. For example, maybe you want to know the answer to 3 ÷ 4. Following the method I show you earlier, first draw a number line from 0 to 3. Then split it into four equal parts. Unfortunately, none of these splits has landed on a number. That's not a mistake. You just have to add some new numbers to the number line, as you can see in Figure 1-19.

Figure 1-19: Fractions on the number line.



Welcome to the world of *fractions*. With the number line labeled properly, you can see that the split closest to 0 is ¾. This image tells you that 3 ÷ 4 = ¾.

The similarity of the expression 3 ÷ 4 and the fraction ¾ is no accident. Division and fractions are closely related. When you divide, you cut things up into equal parts, and fractions are often the result of this process. (I explain the connection between division and fractions in more detail in Chapters 9 and 10.)

Discovering the space in between: Fractions

Fractions help you fill in a lot of the spaces on the number line that fall between the counting numbers. For example, Figure 1-20 shows a close-up of a number line from 0 to 1.

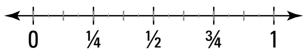


Figure 1-20: Number line depicting some fractions from 0 to 1.

This number line may remind you of a ruler or a tape measure, with lots of tiny fractions filled in. And in fact, rulers and tape measures really are portable number lines that allow carpenters, engineers, and savvy do-it-yourselfers to measure the length of objects with precision.

The addition of fractions to the number line expands the set of integers to the set of *rational numbers*. I discuss the rational numbers in greater detail in Chapter 25.

In fact, no matter how small things get in the real world, you can always find a tiny fraction to approximate it as closely as you need. Between any two fractions on the number line, you can always find another fraction. Mathematicians call this trait the *density* of fractions on the real number line, and this type of density is a topic in a very advanced area of math called *real analysis*.

Four Important Sets of Numbers

In the preceding section, you see how the number line grows in both the positive and negative directions and fills in with lots of numbers in between. In this section, I provide a quick tour of how numbers fit together as a set of nested systems, one inside the other.

When I talk about a *set* of numbers, I'm really just talking about a group of numbers. You can use the number line to deal with four important sets of numbers:

- **Counting numbers (also called natural numbers):** The set of numbers beginning 1, 2, 3, 4 ... and going on infinitely
- **Integers:** The set of counting numbers, zero, and negative counting numbers
- **Rational numbers:** The set of integers and fractions
- **Real numbers:** The set of rational and irrational numbers

The sets of counting numbers, integers, rational, and real numbers are nested, one inside another. This nesting of one set inside another is similar to the way that a city (for example, Boston) is inside a state (Massachusetts), which is inside a country (the United States), which is inside a continent (North America). The set of counting numbers is inside the set of integers, which is inside the set of rational numbers, which is inside the set of real numbers.

Counting on the counting numbers

The set of *counting numbers* is the set of numbers you first count with, starting with 1. Because they seem to arise naturally from observing the world, they're also called the *natural numbers:*

1 2 3 4 5 6 7 8 9 ...

The counting numbers are infinite, which means they go on forever.

When you add two counting numbers, the answer is always another counting number. Similarly, when you multiply two counting numbers, the answer is always a counting number. Another way of saying this is that the set of counting numbers is *closed* under both addition and multiplication.

Introducing integers

The set of *integers* arises when you try to subtract a larger number from a smaller one. For example, 4 – 6 = –2. The set of integers includes the following:

- The counting numbers
- Zero
- The negative counting numbers

Here's a partial list of the integers:

... –4 –3 –2 –1 0 1 2 3 4 ...

Like the counting numbers, the integers are closed under addition and multiplication. Similarly, when you subtract one integer from another, the answer is always an integer. That is, the integers are also closed under subtraction.

Staying rational

Here's the set of *rational numbers*:

- Integers
  - Counting numbers
  - Zero
  - Negative counting numbers
- Fractions

Like the integers, the rational numbers are closed under addition, subtraction, and multiplication. Furthermore, when you divide one rational number by another, the answer is always a rational number. Another way to say this is that the rational numbers are closed under division.

Getting real

Even if you filled in all the rational numbers, you'd still have points left unlabeled on the number line. These points are the irrational numbers.

An *irrational number* is a number that's neither a whole number nor a fraction. In fact, an irrational number can only be approximated as a *non-repeating decimal*. In other words, no matter how many decimal places you write down, you can always write down more; furthermore, the digits in this decimal never become repetitive or fall into any pattern. (For more on repeating decimals, see Chapter 11.)

The most famous irrational number is π (you find out more about π when I discuss the geometry of circles in Chapter 17.):

$$\pi = 3.14159265358979323846264338327950288419716939937510 \ldots$$

Together, the rational and irrational numbers make up the *real numbers,* which comprise every point on the number line. In this book, I don't spend too much time on irrational numbers, but just remember that they're there for future reference.