Criteria for evaluating estimators

→Unbiased Estimator

→UMVUE

Unbiased Estimator

An estimator T is said to be an unbiased estimator of $\tau(\theta)$ if $E(\mathrm{T}) = \tau(\theta)$ for all $\theta \in \Omega$. Otherwise we say that T is a biased estimator of $\tau(\theta)$

Example 1

If $x_1, x_2, \ldots, x_n$ denote a random variable of size n from $f(x)$ with $E[X] = \mu$ and $V(X) = \sigma^2$ then $E[s^2] = \sigma^2$

Example 2

$X_i \sim EXP(\theta)$

proof that $\hat{\theta} = \bar{x}$

Obvious question….?

Which estimators are "best" in some sense?

→Idea :

We've select the estimator that tend to be closest or "most" concentrated around the true value of the parameter

Say that

T_1 is more concentrated than T_2 about $\tau(\theta)$ if

$$P[\tau(\theta)-\varepsilon < T_1 < \tau(\theta)+\varepsilon], \quad \forall \varepsilon > 0$$

An estimator is most concentrated if it is more concentrated than any other estimator

So, it follows from the Chebychev inequality…

T is an unbiased estimator of $\tau(\theta)$,

$$P[\tau(\theta) - \varepsilon < T < \tau(\theta) + \varepsilon] \geq 1 - \frac{\text{var}(T)}{\varepsilon^2}, \quad \forall \varepsilon > 0$$

Example 3

If $\hat{\theta}_1 = \bar{X}$ and $\hat{\theta}_2 = n\bar{X}_{1:n}$ then both estimators are biased for θ

but $\text{V}(\hat{\theta}_1) = \dfrac{\theta^2}{n}$ and $\text{V}(\hat{\theta}_2) = \theta^2$

Which better estimator?

UMVUE

If $X_1, X_2, \ldots, X_n$ denote a random variable of size n from $f(x;\theta)$.

An estimator T^* of $\tau(\theta)$ is called a

Uniformly Minimum Variance Unbiased Estimator (UMVUE)

of $\tau(\theta)$ if :

1. T^* is unbiased for $\tau(\theta)$
2. for any unbiased estimator T of $\tau(\theta)$, $\mathrm{V}(T^*) \leq \mathrm{V}(T)$, $\forall \theta \in \Omega$

CRLB
Cramer Rao - Lower Bound

T is an unbiased estimator of $\tau(\theta)$, then CRLB based on a sample random is

$$V(T) \geq \frac{\left[\tau'(\theta)\right]^2}{nE\left[\frac{\partial}{\partial\theta}\ln f(X;\theta)\right]^2}$$

Example 4

If $X_i \sim EXP(\theta)$

then find CRLB for θ!

Efficiency

The relative efficiency of an unbiased estimator T of $\tau(\theta)$ to another unbiased estimator T^* of $\tau(\theta)$ is given by

$$\mathrm{re}\left(T, T^*\right) = \frac{V\left(T^*\right)}{V(T)}$$

An unbiased estimator T^* of $\tau(\theta)$ is said to be efficient if $\mathrm{re}\left(T, T^*\right) \leq 1$ for all unbiased estimators T of $\tau(\theta)$, $\forall\, \theta \in \Omega$

The efficiency of an unbiased estimator T of $\tau(\theta)$ is given by:

$$\mathrm{e}(T) = \mathrm{re}\left(T, T^*\right)$$

if T^* is an efficient estimator of $\tau(\theta)$

Example 5

consider $\hat{\theta}_1 = \overline{X}$ and $\hat{\theta}_2 = n\overline{X}_{1:n}$ are both estimators are biased for θ

with $\mathrm{V}\left(\hat{\theta}_1\right) = \dfrac{\theta^2}{n}$ and $\mathrm{V}\left(\hat{\theta}_2\right) = \theta^2$

Which one estimator that an efficient estimator for θ?

Definition

If T is an estimator of $\tau(\theta)$ then the bias is given by

$$b(T) = E[T] - \tau(\theta)$$

the Mean Squared Error (MSE) of T is given by:

$$MSE(T) = E[T - \tau(\theta)]^2$$

Theorem

If T is an estimator of $\tau(\theta)$ then $MSE(T) = V(T) + [b(T)]^2$