Teknik Pengintegralan
Bagian 1

1. Substitusi
2. Integral Fungsi Trigonometri
3. Substitusi Rasional

1. Substitusi

Prosedur :

$$\int \underbrace{f(x)\,dx}_{h(u)du} \quad \Rightarrow \text{suppose } u = g(x)$$

$$\int f(x)\,dx = \int h(u)du = H(u) + C = H(g(x)) + C$$

Ex

1. Tentukan $\int \left(2x\sqrt{4x^2+5}\right)dx$

2. Jika $\int_1^2 f(x)dx = 6$ **Maka dengan menggunakan teknik substitusi, hitung :**

$$\int_0^1 xf\left(x^2+1\right)dx$$

3. Hitung

$$\int \frac{e^x}{1+e^x}dx$$

2. Integral Trigonometri

Ingat bentuk dasar :

1. $\sin^2 x + \cos^2 x = 1$

2. $1 + \tan^2 x = \sec^2 x$

3. $1 + \cot^2 x = \operatorname{cosec}^2 x$

4. $\sin 2x = 2 \sin x \cos x$

5. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

6. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

7. $\sin x \sin y = -\frac{1}{2}[\cos(x+y) - \cos(x-y)]$

8. $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

9. $\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$

Rumus Dasar

1. $\int \frac{d}{dx}[f(x)]\,dx = f(x) + C$

2. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$

3. $\int \frac{1}{x}\,dx = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \frac{a^x}{\ln a} + C, \; a > 0, a \neq 1$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x \, dx = \sin x + C$

8. $\int \tan x \, dx = -\ln |\cos x| + C$

$= \ln |\sec x| + C$

9. $\int \cot x \, dx = \ln |\sin x| + C$

10. $\int \sec x \, dx = \ln |\sec x + \tan x| + C$

11. $\int \operatorname{cosec} x \, dx = \ln \left| \operatorname{cosec} x - \cot an x \right| + C$

12. $\int \sec^2 x \, dx = \tan x + C$

13. $\int \operatorname{cosec}^2 x \, dx = -\cot an x + C$

14. $\int \sec x \tan x \, dx = \sec x + C$

15. $\int \operatorname{cosec} x \cot an x \, dx = -\operatorname{cosec} x + C$

16. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$$

17. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arcsin \frac{x}{a} + C$$

18. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} arc\sec \frac{x}{a} + C$$

19. $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$

20. $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$$

21. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)+C$$

22. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)+C$$

23. $$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}x\sqrt{a^2-x^2}+\tfrac{1}{2}a^2\arcsin\frac{x}{a}+C$$

24. $$\int \sqrt{x^2+a^2}\,dx = \tfrac{1}{2}x\sqrt{x^2+a^2}$$

$$+\tfrac{1}{2}a^2\ln\left(x+\sqrt{x^2+a^2}\right)+C$$

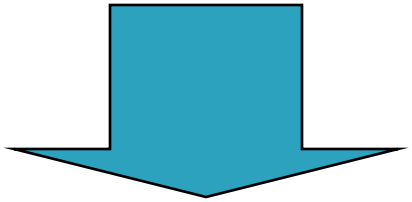
25. $$\int \sqrt{x^2-a^2}\,dx = \tfrac{1}{2}x\sqrt{x^2-a^2}$$

$$-\tfrac{1}{2}a^2\ln\left|x+\sqrt{x^2-a^2}\right|+C$$

Teknik Integrasi dengan fungsi Trigonometri

1. $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

Dengan n bilangan asli positif ganjil



$$\sin^2 x + \cos^2 x = 1$$

2. $\int \sin^n x\,dx$ or $\int \cos^n x\,dx$

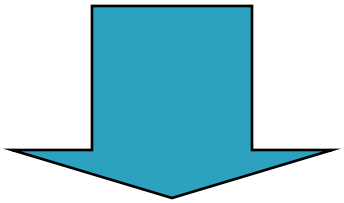
Dengan n bilangan asli positif genap

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

3. $\int \sin^n x \cos^m x \, dx$

Dengan m atau n adalah bilangan asli positif



$$\sin^2 x + \cos^2 x = 1$$

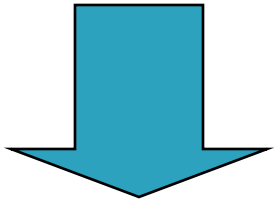
4. $\int \sin^n x \cos^m x \, dx$

Dengan m dan n adalah bilangan asli positif genap

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

5. $\int \tan^n x\,dx$ or $\int \cot an^n dx$



$$\cot an^2 x = \cos ec^2 x - 1$$

$$\tan^2 x = \sec^2 x - 1$$

6. $\int \tan^m x \sec^n x \; dx$

or $\int \cot an^m \cos ec^m x \quad dx$

$\tan^2 x = \sec^2 x - 1$

$\cot an^2 x = \cos ec^2 x - 1$

Example

1. $\int \cos^3 x\, dx$

2. $\int \cos^4 x \sin^3 x dx$

3. $\int \sin 3x \sin 2x dx$

4. $\int \tan^4 x dx$

5. $\int \tan^3 x \sec^4 x dx$

Integrasi dengan subsitusi

$\sqrt{a^2 - x^2}$ → x= a sin t

$\sqrt{a^2 + x^2}$ → x= a tan t

$\sqrt{x^2 - a^2}$ → x= a sec t

Rumus 1

$$\text{substitusi } x = a\sin\varphi \Rightarrow x^2 = a^2\sin^2\varphi$$

$$a^2 - x^2 = a^2 - a^2\sin^2\varphi$$

$$= a^2\left(1 - \sin^2\varphi\right)$$

$$= a^2\cos^2\varphi$$

Contoh

$$1.\int \frac{\sqrt{a-x}}{x^{3/2}}dx$$

$$2.\int \frac{dx}{\left(4-x^2\right)^{3/2}}$$

$$3.\int \frac{dx}{x^2\sqrt{9-x^2}}$$

Rumus 2

Substitusi $x = \frac{a}{b}\tan\varphi$

$$dx = \frac{a}{b}\sec^2\varphi\, d\varphi$$

cth.

1. $\int \frac{dx}{\left(9+x^2\right)^2}$

2. $\int \frac{dx}{x\sqrt{9+4x^2}}$

Rumus3

substitusi $x = \frac{a}{b}\sec\varphi$

$$dx = \frac{a}{b}\sec\varphi\tan\varphi\, d\varphi$$

Ex

$$\int \frac{x}{\sqrt{x^2 - 16}}\, dx$$

Rumus 4

substitusi $u^n = ax + b$

$$nu^{n-1}du = a\,dx$$

$$dx = \frac{nu^{n-1}}{a}du$$

Ex

1. $\int \frac{dx}{(x-2)\sqrt{x+2}}$

2. $\int \frac{dx}{3+\sqrt{x+2}}$

Rumus 5

substitusi $(u-x)^2 = x^2 + ax + b$

$$u^2 - 2ux + x^2 = x^2 + ax + b$$

$$u^2 - 2ux = ax + b$$

$$u^2 - b = ax + 2ux$$

$$x = \frac{u^2 - b}{2u + a}$$

contoh

$$\int \frac{dx}{x\sqrt{x^2 + x + 2}}$$

Rumus 6

substitusi $(p+x)^2 u^2 = b + ax - x^2$

atau

substitusi $(q-x)^2 u^2 = b + ax - x^2$

Ex

$$\int \frac{x}{\left(5 - 4x - x^2\right)^{3/2}} dx$$

Latihan

1. $\int \frac{(x-3)^2}{(6x-x^2)^{3/2}}$

2. $\int \frac{4-x}{\sqrt{4-9x^2}}\,dx$

3. $\int_1^4 \frac{1}{x+\sqrt{x}}dx$

4. $\int \frac{\ln x\sqrt{(\ln x)^2+1}}{x}dx$