# THE EFFECT OF VACCINATION ON THE DISEASE OUTBREAK USING *SIR* ENDEMIC MODEL

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ABSTRACT. In recent time, the vaccination program is a powerful method to control the disease outbreak. The diseases outbreak can be studied using mathematical model. In order to study the disease outbreak, Kermack and McKendrick developed a standard mathematical model . The model is well known as $SIR$ (Susceptible-Infected-Recovered) epidemic model.

In this paper, we will re-derive the *SIR* model including the vital dynamic (birth and death). In order to prevent and control the disease outbreak, we also include the vaccination factor into the model. As the results, the model has two equilibrium points, disease free and endemic equilibrium. The qualitative analysis reveals the vaccination reproductive number $R_v = \frac{\alpha(1-\sigma)}{\mu+\beta}$ and the minimum level of vaccination needed to prevent the disease outbreak $\sigma_c = 1 - \frac{\beta+\mu}{\alpha}$. To illustrate the model, we study an example from Makinde [3]. We compute an approximate solution of the non-linear system of differential equations governing the problem. Graphical result are presented and discussed qualitatively to illustrate the solution.

***Keyword:*** *mathematical model, vaccination, stability, disease outbreak, SIR model.*

## 1. INTRODUCTION

Measles, mumps, rubella, and poliomyelitis are very dangerous diseases. Kinbaby [2] states that these diseases can cause paralysis and death. They are more often infectious children than adults because the children's immunity is more vulnerable than the adults's. The United Nations Children's Fund (UNICEF) [4] states that more than 30.000 children die because of the measles attack in Indonesia. World Health Organization (WHO) [6] states that more than 242.000 children in the world die because of the measles attack. On the other hand, UNICEF [5] states that paralysis is suffered by about 302 children in Indonesia.

The development of science and technology has a main role to control the diseases outbreak. In order to control the diseases outbreak, UNICEF holds a vaccination program in the world. WHO [6] states that the vaccination can

decrease the number of paralysis and death about 68%. This reality shows that the vaccination program is a powerful method to control the diseases outbreak.

The disease outbreak can be studied using mathematical model called $SIR$ epidemic model. In this paper, we include the vital dynamic (birth and death) into the model because the diseases can spread more than a year. We also include the vaccination factor into the model in order to prevent and control the diseases outbreak.

Makinde [3] states that the $SIR$ model is a non-linear system of differential equations. It is hard to determine the exact solution of the system. So, in this paper we compute an approximate solution of the system. In order to know the behaviour of the disease outbreak, we investigate the stability on the equilibrium point of the system.

## 2. Model Construction

In this section, we will re-derive $SIR$ endemic model with vital dynamic and vaccination factor according to Makinde [3]. In this model, we assume that the outbreak happens in a closed population. The number of total population is constant and denoted as $N$. The incubation period of the disease is ignored.

According to Hethcote [1], the $SIR$ epidemic model developed by Kermack and McKendrick is

$$\begin{aligned} \frac{dS}{dt} &= -\alpha S\frac{I}{N} \\ \frac{dI}{dt} &= \alpha S\frac{I}{N} - \beta I \\ \frac{dR}{dt} &= \beta I \end{aligned} \tag{2.1}$$

where $\alpha$ is a contact rate and $\beta$ is a removal rate. In this model, population is classified into susceptible, infected, and recovered class.

(1) Susceptible class contains individuals who are healthy and infectible.
(2) Infected class contains individuals who are infected and are able to transmit the disease.
(3) Recovered class contains individuals who are immune.

We denote the number of individuals in susceptible, infected, and recovered class at time $t$ respectively as $S(t)$, $I(t)$, and $R(t)$.

Now, we will include the vital dynamic and vaccination factor into the model. The vital dynamic contains the rate of birth and date. We assume that the birth rate is equal to the death rate and we denote the birth or death rate as $\mu$.

The number of births is proportional to the number of total population $N$. So, the number of births can be expressed by $\mu N$. We assume that all births are susceptible. The number of death on each class is proportional to the number of population in each class. So, the number of death on susceptible, infected, and recovered class respectively can be expressed as $\mu S$, $\mu I$, $\mu R$. Now, we can obtain

$$\mu S + \mu I + \mu R = \mu(S + I + R) = \mu N. \tag{2.2}$$

Equation (2.2) shows that the number of births is equal to the number of deaths.

Then, we set the vaccination level or the fraction of population vaccinated at birth each year as $\sigma$. The number of population vaccinated is proportional to the number of births and can be expressed by $\sigma\mu N$. The vaccinated individuals will recover and then enter the recovered class. We assume that the efficacy of the vaccine is 100% so the vaccinated population can not be infected again. The number of unvaccinated births

$$\mu N - \sigma\mu N = (1 - \sigma)\mu N$$

will enter the susceptible class. So, we can obtain

$$\begin{aligned} \frac{dS}{dt} &= (1-\sigma)\mu N - \alpha\frac{SI}{N} - \mu S \\ \frac{dI}{dt} &= \alpha\frac{SI}{N} - \beta I - \mu I \\ \frac{dR}{dt} &= \sigma\mu N + \beta I - \mu R \end{aligned} \tag{2.3}$$

with the initial condition $S(0) > 0, I(0) > 0$ and $R(0) \geq 0$. The summary of the dynamical population of the system (2.3) is shown by Figure 1.

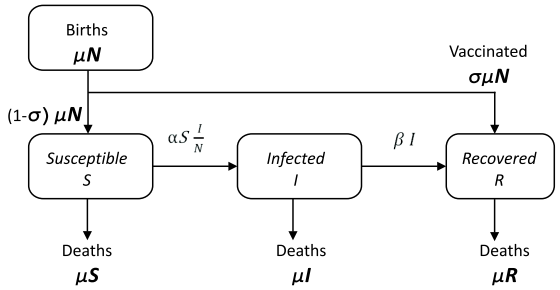


Figure 1. The dynamical population of the system (2.3)

In order to simplify the system (2.3), we scale system (2.3) by the number of total population $N$. We denote the fraction of the number of population on

each class as $s$, $i$, and $r$. So, we obtain

$$\begin{aligned} \frac{ds}{dt} &= (1-\sigma)\mu - \alpha si - \mu s \\ \frac{di}{dt} &= \alpha si - \beta I - \mu i \\ \frac{dr}{dt} &= \sigma\mu + \beta i - \mu r. \end{aligned} \tag{2.4}$$

The system (2.4) is a nonlinear system of $SIR$ endemic model with vital dynamic and vaccination factors.

## 3. Equilibrium Point

On the system (2.4), variable $r$ does not appear on the first and second equations. So, the number of population in susceptible and infected classes are independent with the number of population on recovered class. On the other hand, we have $s + i + r = 1$. Using these conditions, we can simplify the system (2.4) by

$$\begin{aligned} \frac{ds}{dt} &= (1-\sigma)\mu - \alpha si - \mu s \\ \frac{di}{dt} &= \alpha si - \beta I - \mu i. \end{aligned} \tag{3.1}$$

There are two equilibrium points of system (3.1).

(1) Disease free equilibrium (DFE) point $E_0 = \big(s_0, i_0\big) = \big((1-\sigma), 0\big)$.
The value $i_0 = 0$ means that there is no individual in infected class. So, the disease can not spread.

(2) Endemic equilibrium point

$$E_e = \big(s_e, i_e\big) = \big(\frac{\mu+\beta}{\alpha}, \frac{(1-\sigma)\mu\alpha - \mu(\mu+\beta)}{(\mu+\beta)\alpha}\big).$$

The non zero value of $i_e$ means that there are individuals on infected class so the disease can spread.

## 4. Vaccination Reproduction Number

In order to know the level of the disease outbreak, we need a parameter called a vaccination reproduction number. According to Hethcote [1], the vaccination reproduction number is the expected number of secondary cases produced in a completely susceptible population by a typical infective individual.

The vaccination reproductive number can be expressed as

$$R_v = \frac{\alpha(1-\sigma)}{\mu+\beta}. \tag{4.1}$$

According to Makinde [3], from equation (4.1), we can define the minimum level of the vaccination needed in order to prevent the disease outbreak as

$$\sigma_c = 1 - \frac{\beta + \mu}{\alpha}. \tag{4.2}$$

In order to successfully prevent disease, Makinde [3] states that the vaccination level $\sigma$ should be larger than $\sigma_c$.

## 5. Stability

In this section, we determine the stability on the equilibrium point by using the eigenvalues of Jacobian matrix from the system (3.1). The Jacobian matrix is obtained by linearize the system (3.1) using a Taylor expansion. The Jacobian matrix of the system (3.1) is

$$J = \begin{pmatrix} -\alpha i - \mu & -\alpha s \\ \alpha i & \alpha s - (\beta + \mu) \end{pmatrix}. \tag{5.1}$$

The eigenvalues of $J$ on the DFE point are $\lambda_1 = -\mu$ and $\lambda_2 = \alpha(1-\sigma) - (\beta + \mu)$. The DFE point will asymptotically stable if $\lambda_{1,2} < 0$. We know that $\mu > 0$, so $\lambda_1 < 0$. If $\lambda_2 < 0$ then $\alpha(1-\sigma) < (\beta + \mu)$ and

$$\frac{\alpha(1-\sigma)}{(\beta + \mu)} = R_v < 1.$$

So, the DFE point will asymptotically stable if $R_v < 1$. In the other hand, if $R_v > 1$ then DFE point will unstable.

The endemic equilibrium $E_e$ point can be expressed in term of vaccination reproduction number $R_v$

$$(s_e, i_e) = \big(\frac{1-\sigma}{R_v}, \frac{\mu}{\alpha}(R_v - 1)\big). \tag{5.2}$$

Using equation (5.2), the eigenvalues of Jacobian matrix on the endemic equilibrium point are

$$\lambda_{1,2} = -\frac{\mu}{2} R_v \pm \frac{1}{2}\sqrt{\mu^2 R_v^2 - 4(R_v - 1)\mu(\beta + \mu)}.$$

From equation (5.2), the endemic equilibrium point will appear for $R_v > 1$. The eigenvalues $\lambda_{1,2}$ are negative real or complex numbers with negative real part if

$$1 < R_v < \frac{4(\beta + \mu)}{\mu}. \tag{5.3}$$

So, the condition (5.3) gives a consequence that the endemic equilibrium point will asymptotically stable.

## 6. Example

As an example, we will study the disease outbreak given by Makinde [3]. In this example, we use the contact rate $\alpha = 0.8$, the removal rate $\beta = 0.03$, and the birth or death rate per year $\mu = 0.4$. Then, the initial condition, the fraction of population on each class are $s(0) = 0.8$, $i(0) = 0.2$, and $r(0) = 0$. If there is no vaccination program to control the outbreak, then the behaviour of the outbreak is shown by the Figure 2 (left). The Figure 2 (left) shows that the

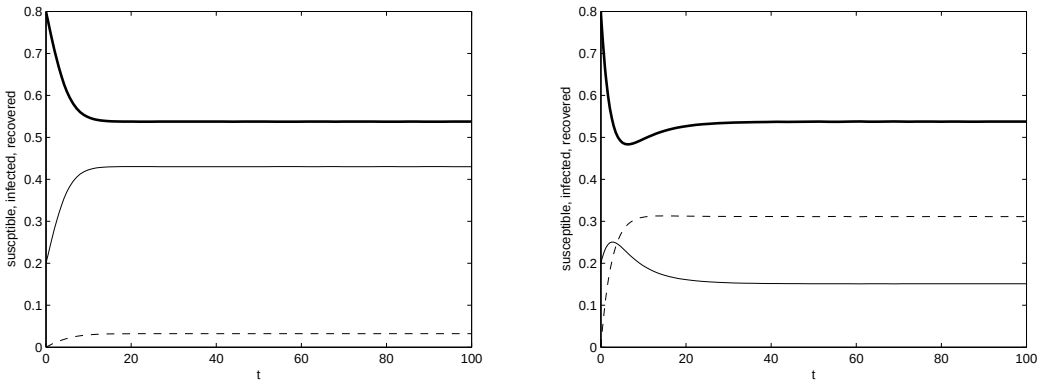


Figure 2. The fraction of susceptible (thick line), infected (thin line), and recovered (dash line) for $\sigma = 0$ (left) and $\sigma = 0.3$ (right)

disease still outbreaks for a long time. Theoritically, using equation (4.1), we can obtain $R_v = 1.86$. So, the disease will become endemic. On this case, the model has one equilibrium point. This point is endemic equilibrium point $(s_e, i_e) = (0.5375, 0.4302)$. This equilibrium is stable because it fulfills equation (5.3).

In order to prevent and control the disease, we hold a vaccination program. Then, If we set the fraction of population vaccinated at birth each year $\sigma = 0.3$, then the behaviour of the outbreak is shown by the Figure 2 (right). The Figure 2 (right) shows that the disease also still outbreak for a long time and become endemic. So, we conclude that the vaccination is not enough to prevent the disease. For $\sigma = 0.3$, the model has one equilibrium point called endemic equilibrium point. The value of $R_v$ is 1.3. So, the endemic equilibrium is stable.

In order to describe a treatment for preventing the disease successfully, Makinde [3] states that the vaccination program must be larger enough than the minimum level of the vaccination. We use equation (4.2) to obtain the minimum level of the vaccination needed $\sigma_c = 0.4625$. Now, we interest to examine the effect of vaccination on the infected class. Our goal is to make the disease dies out. For $\sigma = \sigma_c$, the behaviour of the outbreak is shown by the Figure 3

(left). The Figure 3 (left) shows that the disease will die out. But, we need more

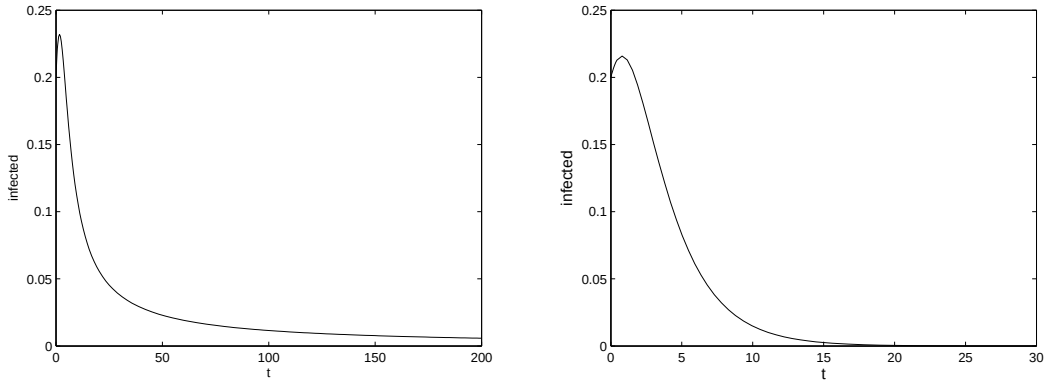


Figure 3. The fraction of infected individuals for $\sigma = \sigma_c$ (left) and $\sigma = 0.9$ (right)

than 200 years to make the disease dies out. So, it is not efficient if we hold a vaccination program at level $\sigma_c$. Then, we examine the behaviour for $\sigma > \sigma_c$. For example, we set $\sigma = 0.9$. The behaviour of the outbreak is shown by the Figure 3 (right). The Figure 3 (right) shows that the disease will die out for about 20 years. So, if we choose to hold a vaccination program at level $\sigma = 0.9 > \sigma_c$, we obtain a better result than $\sigma_c$.

The stability of the equilibrium points can be determined using the trajectories as shown by Figure 4. The Figure 4 (left) shows the trajectory for $\sigma = 0.3$.

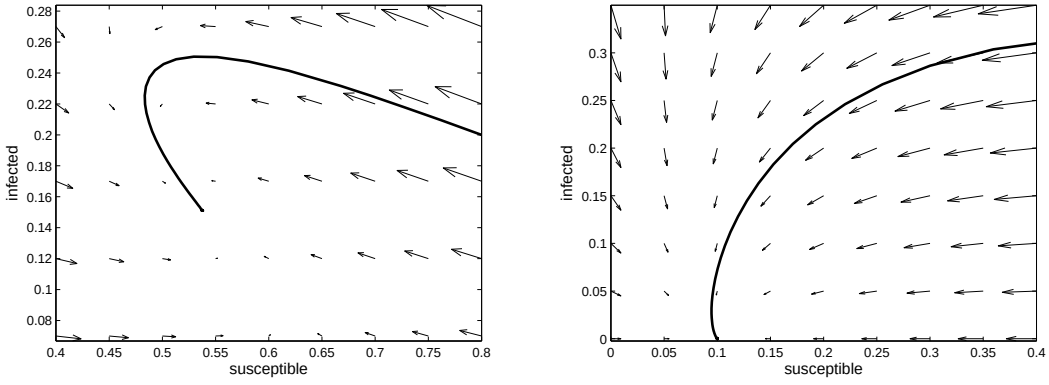


Figure 4. The trajectory of the model for $\sigma = 0.3$ (left) and $\sigma = 0.9$ (right)

The endemic equilibrium point for $\sigma = 0.3$ is stable because the solution goes to the direction of equilibrium point. The Figure 4 (right) shows the trajectory for $\sigma = 0.9$. The DFE point for $\sigma = 0.9$ is stable because the solution goes to the direction of equilibrium point.

## 7. Conclusion

Based on our observation, we make four conclusions.

(1) The $SIR$ endemic model with vaccination program can be expressed as

$$\frac{ds}{dt} = (1-\sigma)\mu - \alpha si - \mu s$$
$$\frac{di}{dt} = \alpha si - \beta i - \mu i$$
$$\frac{dr}{dt} = \sigma\mu + \beta i - \mu r.$$

(2) This model has two equilibrium points

$$E_0 = \big((1-\sigma), 0\big) \text{ and } E_e = \big(\frac{\mu+\beta}{\alpha}, \frac{(1-\sigma)\mu\alpha - \mu(\mu+\beta)}{(\mu+\beta)\alpha}\big).$$

The points $E_0$ and $E_e$ will asymptotically stable respectively if $R_v < 1$ and $1 < R_v < \frac{4(\beta+\mu)}{\mu}$.

(3) In order to prevent the disease successfully, the minimum level of the vaccination needed is

$$\sigma_c = 1 - \frac{\beta+\mu}{\alpha}.$$

(4) In this example, the equilibrium point is asymptotically stable and the minimum level of the vaccination needed is $\sigma = 0.4625$.

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